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Dynamics and Stability of a Freely Precessing Spacecraft Containing a Nutation Damper

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The stability of a torque-free inertially symmetric spinning spacecraft containing a partially filled viscous ring nutation damper and the performance of this damper on the removal of the wobble motion of the spacecraft are investigated. The motion behavior about three different relative equilibria are studied. The equations for stability conditions and decay time constant for these three relative equilibria are obtained in explicit form. The effects of parameters of the spacecraft and damper on the stability criteria and decay time constant are quite different for these three equilibria. Numerical solutions show that when the motion about one of these three relative equilibria loses its stability, it converges to either one of the other two, depending on which stability criterion is satisfied.

Introduction

GENERALLY speaking, there are two purposes for spacecraft to be equipped with nutation dampers. The first one is to stabilize the spacecraft whose attitude motion is unstable with the nutation damper removed (see, e.g., Tonkin[1] and Annett et al.[2]). The second purpose is to consider the nutation damper as a passive positioning control device, since it can dissipate the energy of the coning motion of the spacecraft and convert its transverse angular momentum into the spin angular momentum so as to confine the spacecraft to be close to the pure spin state. The nutation damper considered in this paper is for the second purpose. The addition of a nutation damper to the spacecraft that is stabilized by spin might cause an alteration in the stability criteria, for example, the parameters of the damper may appear in the equations for stability conditions. Thus, the paper studies the attitude stability of a spacecraft with and without nutation dampers, in addition to the study of the damping performance.

The damping devices for removing the coning motion are of various types. There are ball-in-tube dampers,[3] eddy current spherical magnetically anchored dampers,[4] viscous spherical nutation dampers,[5,6] elastomer dampers,[7,8] and viscous ring nutation dampers.[9–12] In this paper a ring damper partially filled with viscous fluid, as used in Refs. 10–12, is considered. The spacecraft considered here is assumed to be an inertially symmetric rigid body, that is, two of its principal moments of inertia are equal. This body does not necessarily have to be a body of revolution. A rectangular parallelepiped may possess such inertial symmetry. For the sake of convenience, however, it is shown as a body of revolution, such as a cylinder, in the subsequent figures. In this paper the ring damper is mounted on a plane perpendicular to the axis of revolution (or the axis of inertial symmetry). The spin rate of a spin-stabilized spacecraft is usually low. For a rotor spinning at low speed, the shape of the viscous fluid in the ring looks like a crescent.[11] Thus we model the viscous fluid as a rigid slug and assume that the frictional force between the slug and the wall of the ring is linearly proportional to the relative velocity between them.

From the attitude equations of motion for the aforementioned system, it was found that there are three relative equilibria denoted by E_1, E_2, and E_3. At the state of relative equilibrium E_1 the spacecraft spins about the inertially symmetric axis and precesses about the constant angular momentum vector at a constant nutational angle. At E_2 and E_3, the spacecraft spins about the axis perpendicular to the inertially symmetric axis, that is, the spin axis is parallel to the plane of the ring damper. The spinning spacecraft is in coning motion with constant precession rate and nutational angle at E_2 but in pure spin without precession at E_3. References 9 and 12 adopted the method of energy-sink approximation to estimate the decay time constant of the nutation angle. This method considers the nutation angle as a parameter rather than a dependent variable and simply assumes that the energy dissipation rate of the fluid equals the rate of kinetic energy change of the rotor. Therefore, it circumvents the need to establish and solve the coupled equations of motion of the rotor and fluid. References 2, 6, and 10 used zero-order approximation to obtain the solution of the spacecraft, and in doing so the result of constant spin rate was ascertained. The decay time constant was obtained by solving the first-order approximate damper equations. In this paper we solve the coupled equations of motion directly. Therefore, the assumption of quasi-steady nutational motion and the restriction of constant spin rate are removed. In doing so the analysis is more rigorous and more information might be obtained. Alfriend[10] analyzed the decay time constant for the relative equilibrium E_1 based on the aforementioned approximate method; here we further study the behavior of motion about the E_2 and E_3, which has not been done before, in a rigorous manner.

The reason that nonlinear equations of motion are analyzed is because the attitude equations are nonlinear in nature, and thus the behavior of nutational motion that is not limited to be small can be understood, and because the linearized equations about any of these three relative equilibria have one zero eigenvalue, which causes a linear stability analysis to be inconclusive. Central to the analysis in this paper is the use of the center manifold theorem to remove the stable subspace and determine the stability and asymptotic behavior for the flow in the center manifold. The analytical results agree very well with the numerical results that are obtained by integrating the original nonlinear equations numerically. The behavior of the system losing its stability about one equilibrium and moving toward another one is revealed by numerical results and is shown in graphic form.

Equations of Motion

Referring to Fig. 1, the coordinate system OXYZ is the inertial frame of reference, with the origin O fixed in space coinciding with the center of mass of the inertially symmetric spacecraft. The local coordinate system $oxyz$ is fixed on the spacecraft so that the z axis is the axis of inertial symmetry and the origin o coincides with the

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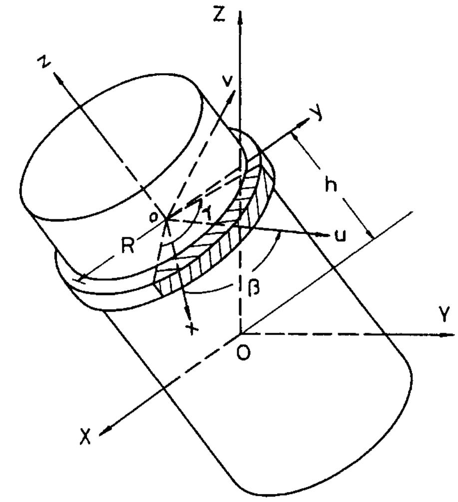

Fig. 1 Mathematical model and coordinate systems.

geometric center of the damper ring. The viscous fluid in the ring is assumed to be a rigid slug and the angle of fill is denoted by γ. The radius of the ring, the mass of the fluid, and the height of the ring to the center of mass of the spacecraft are R, m, and h, respectively. Gravity effects are neglected. The mass of the viscous fluid is assumed to be much smaller than that of the spacecraft, and both mR^2 and mh^2 are also much smaller than the principal moments of inertia of the spacecraft. The coordinate system $ouvz$ is fixed on the fluid slug, with the u axis passing through the center of mass of the slug and making an angle β counterclockwise with the x axis. Let $\boldsymbol{e}_u$, $\boldsymbol{e}_v$, and $\boldsymbol{e}_z$ denote the unit vectors along the u, v, and z axes and ω_u, ω_v, and ω_z denote the components of angular velocity of the spacecraft along the $ouvz$ coordinate system. Then, the angular velocity of the slug is $\underline{\omega}_s = \omega_u \boldsymbol{e}_u + \omega_v \boldsymbol{e}_v + (\omega_z + \dot{\beta})\boldsymbol{e}_z$. The total angular momentum about the origin O of the $Ouvz$ coordinate system is

$$\boldsymbol{h} = [(A + I_u)\omega_u - I_{uz}(\omega_z + \dot{\beta})]\boldsymbol{e}_u + (A + I_v)\omega_v \boldsymbol{e}_v + [C\omega_z + I_z(\omega_z + \dot{\beta}) - I_{uz}\omega_u]\boldsymbol{e}_z \tag{1}$$

where A and C are the symmetric and transverse principal moments of inertia of the spacecraft and where I_u, I_v, I_z, and I_{uz}, the nonzero entries of the inertia matrix of the slug, are

$$\begin{aligned} I_u &= m[h^2 + (1 - \sin\gamma/\gamma)R^2] \\ I_v &= m[h^2 + (1 + \sin\gamma/\gamma)R^2] \\ I_z = mR^2, &\qquad I_{uz} = mhR\sin(\gamma/2)/(\gamma/2) \end{aligned} \tag{2}$$

The system is free of external torques; from the conservation of angular momentum, we have

$$\frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} = \left(\frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t}\right)_{Ouvz} + \underline{\omega}_s \times \boldsymbol{h} = 0 \tag{3}$$

which is equivalent to three equations of motion. The other equation of motion corresponding to the generalized coordinate β is obtained by using Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\beta}} - \omega_v \frac{\partial T}{\partial \omega_u} + \omega_u \frac{\partial T}{\partial \omega_v} = Q_\beta \tag{4}$$

where T is the kinetic energy of the system and is in the form

$$T = (A/2)\left(\omega_u^2 + \omega_v^2\right) + (C/2)\omega_z^2 + \tfrac{1}{2}\left[I_u\omega_u^2 + I_v\omega_v^2 + I_z(\omega_z + \dot{\beta})^2 - 2I_{uz}\omega_u(\omega_z + \dot{\beta})\right] \tag{5}$$

and Q_β is the generalized force associated with the coordinate β. It is assumed that the frictional force between the slug and the wall of the ring is proportional to the relative velocity between them. Thus, $Q_\beta = -C_d R^2 \dot{\beta}$, where C_d is the coefficient of friction. To nondimensionalize the equations of motion, we introduce the following dimensionless quantities:

$$\begin{aligned} &\tau = \Omega t, \qquad \omega_u = p\Omega, \qquad \omega_v = q\Omega \\ &\omega_z = r\Omega, \qquad \sigma = C/A, \qquad b = h/R \\ &\eta = C_d/m\Omega, \qquad \epsilon = mR^2/A\bar{\gamma} \end{aligned} \tag{6}$$

where ϵ is a small parameter because it is assumed that the mass of the viscous fluid is much smaller than that of the spacecraft, Ω the spin rate of the spacecraft at its initial pure spin state, and $\bar{\gamma} = \gamma/2\pi$. The substitution of Eqs. (1), (2), (5), and (6) into Eqs. (3) and (4) yields the dimensionless equations of motion as

$$p' + \frac{(\lambda r - \beta')q}{D_1} + \frac{[A_1(r + \beta') + A_2 p]q}{D_1} + \frac{A_3\beta'}{\Omega D_1} = 0 \tag{7a}$$

$$q' - \frac{(\lambda r - \beta')p}{D_2} - \frac{[B_1(r + \beta') + B_2 p]p}{D_2} + \frac{B_2(r + \beta')^2}{D_2} = 0 \tag{7b}$$

$$\beta'' + C_1\beta'/\Omega + C_2 pq + C_3[q(r + \beta') - p'] = 0 \tag{7c}$$

$$r' = C_4\beta'/\Omega \tag{7d}$$

where primes denote differentiation respect to τ, and the parameter λ and the coefficients A_1, A_2, A_3/Ω, B_1, B_2, C_1/Ω, C_2, C_3, C_4/Ω, D_1, and D_2 that are expressed in terms of ϵ are listed in the Appendix. The state (p_e, q_e, r_e, β_e) of relative equilibria can be found by setting $p' = 0$, $q' = 0$, $r' = 0$, and $\beta'' = \beta' = 0$ in Eq. (7), and they satisfy the following equations:

$$[(\lambda + A_1)r_e + A_2 p_e]q_e = 0 \tag{8a}$$

$$(C_3 r_e + C_2 p_e)q_e = 0 \tag{8b}$$

$$B_2 r_e^2 - (\lambda + B_1)r_e p_e - B_2 p_e^2 = 0 \tag{8c}$$

Solving Eq. (8), we obtain three relative equilibria E_1, E_2, and E_3 as

$$E_1: \qquad q_e = 0, \qquad p_e = \frac{-(\lambda + B_1) + \sqrt{(\lambda + B_1)^2 + 4B_2^2}}{2B_2} r_e \tag{9}$$

$$E_2: \qquad q_e = 0, \qquad p_e = \frac{-(\lambda + B_1) - \sqrt{(\lambda + B_1)^2 + 4B_2^2}}{2B_2} r_e \tag{10}$$

$$E_3: \qquad p_e = 0, \qquad r_e = 0, \qquad q_e \neq 0 \tag{11}$$

Together with the equation $|\boldsymbol{h}| = \text{const}$, the nonzero values of p_e, q_e, and r_e in Eqs. (9–11) can be determined. In the rest of the paper we consider that $\lambda = \sigma - 1 \neq \mathcal{O}(\epsilon)$, that is, the spacecraft close to the case of inertial sphere is not considered. From the Appendix we know that both B_1 and B_2 are of order ϵ. For the relative equilibrium E_1, Eq. (9) expanded up to ϵ^1 is

$$p_e = \frac{-1 + \sqrt{1 + \mathcal{L}^2}}{\mathcal{L}} r_e \cong \frac{B_2}{(\lambda + B_1)} r_e \cong \frac{B_2}{\lambda} r_e = -\frac{\epsilon\bar{\gamma}\kappa b}{\lambda} r_e \tag{12}$$

where $\mathcal{L} = 2B_2/(\lambda + B_1)$. The sign of p_e depends on the sign of λ. Equation (12) implies that $r_e = \mathcal{O}(1)$ and $p_e = \mathcal{O}(\epsilon)$. The physical meaning for this relative equilibrium is that the spacecraft spins about the z axis, and the z axis precesses steadily about the angular momentum vector $\boldsymbol{h}$ with a small (or residual) constant nutational angle θ_n, which satisfies

$$\tan\theta_n = \frac{h_t}{h_z} = \frac{(A + I_u)p_e - I_{uz}r_e}{(C + I_z)r_e - I_{uz}p_e} \approx \frac{\epsilon\bar{\gamma}\kappa b}{\sigma - 1} \tag{13}$$

where h_t and h_z are the transverse and z-axis components of the angular momentum vector $\boldsymbol{h}$. For the relative equilibrium E_2, Eq. (10) expanded up to ϵ^1 is

$$r_e = \frac{-\mathcal{L}}{1+\sqrt{1+\mathcal{L}^2}}p_e \cong -\frac{B_2}{\lambda}p_e = \frac{\epsilon\bar{\gamma}\kappa b}{\lambda}p_e \tag{14a}$$

Equation (14a) implies that $p_e = \mathcal{O}(1)$ and $r_e = \mathcal{O}(\epsilon)$. The associated physical meaning is that the spacecraft spins about the u axis and the u axis precesses steadily about the vector $\boldsymbol{h}$ with a constant residual nutation angle that is given by the equation

$$\tan\theta_n = \frac{h_t}{h_z} \approx \frac{\epsilon\bar{\gamma}\kappa b}{1-\sigma} \tag{14b}$$

Equation (11) indicates that the configuration of the relative equilibrium E_3 is that the spacecraft is in pure spin about the v axis and there is no precession and no residual nutational motion.

Nonlinear Analysis

Relative Equilibria E_3 and E_2 Cases

The case of relative equilibrium E_3 is considered first. To study the local motion about E_3, we let $p = x_1$, $q = q_e + x_2$, $r = x_3$, and $\beta' = x_4$. Substituting these into Eq. (7), we have

$$\boldsymbol{x}' = \bar{\boldsymbol{C}}\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}) \tag{15}$$

where

$$\bar{\boldsymbol{C}} = \begin{bmatrix} -\dfrac{A_2}{D_1}q_e & 0 & -\dfrac{\lambda+A_1}{D_1}q_e & -\dfrac{A_3+(A_1-1)q_e\Omega}{\Omega D_1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{C_4}{\Omega} \\ -\left(C_2+\dfrac{A_2}{D_1}\right)q_e & 0 & -\dfrac{\lambda+A_1+D_1}{D_1}C_3q_e & -\dfrac{C_1D_1+A_3C_3+\Omega C_3q_e(A_1+D_1-1)}{\Omega D_1} \end{bmatrix} \tag{16}$$

and the nonlinear vectors $\boldsymbol{f}$ are

$$f_1(\boldsymbol{x}) = -[(\lambda+A_1)x_3 + (A_1-1)x_4 + A_2x_1](x_2/D_1) \tag{17a}$$

$$f_2(\boldsymbol{x}) = [(\lambda+B_1)x_3 + (B_1-1)x_4 + B_2x_1] \times (x_1/D_2) - (B_2/D_2)(x_3+x_4)^2 \tag{17b}$$

$$f_3(\boldsymbol{x}) = 0 \tag{17c}$$

$$f_4(\boldsymbol{x}) = -[(A_2C_3 + C_2D_1)x_1 + (\lambda+A_1+D_1)C_3x_3 + (A_1-1+D_1)x_4](x_2/D_1) \tag{17d}$$

The characteristic equation of the linear part of Eq. (15) is

$$s\left[s^3 - (c_{11}+c_{44})s^2 + (c_{11}c_{44} - c_{14}c_{41} - c_{34}c_{43})s + c_{34}(c_{11}c_{43} - c_{13}c_{41})\right] = 0 \tag{18}$$

where s is the eigenvalue and c_{ij} are the elements of the matrix $\bar{\boldsymbol{C}}$. Equation (18) shows that the linear part of system (15) has one zero eigenvalue no matter what system parameters are varied. It is known from classical dynamic analysis (Ref. 13, pp. 228, 229) that the linearized, rotational equations of motion about one relative equilibrium for a torque-free spinning rigid body always has one zero eigenvalue if the equations are expressed in terms of quasi-coordinates, i.e., the components of angular velocity. The property of possessing one zero eigenvalue remains unchanged even if the viscous ring nutation damper is included. This zero eigenvalue may be removed by using another set of generalized coordinates such as Euler's angle, plus the employment of Routh's method for eliminating the cyclic coordinate; however, it will result in the equations of motion having five first-order differential equations, which is one more in number than the equations expressed in terms of quasi-coordinates, and hence will raise the degree of difficulty for analysis. Because the stability of a nonlinear system (or its asymptotic behavior of solutions) cannot be determined by the linearization if the linearized system has zero or purely imaginary eigenvalues,[14,15] the nonlinear equations must be used in the study of the stability problem.

Because of the complexity of the elements of the matrix $\bar{\boldsymbol{C}}$, it is a formidable task to convert system (15) into an explicit canonical form in the direct use of the matrix $\bar{\boldsymbol{C}}$. Thus, matrix $\bar{\boldsymbol{C}}$ is split into two parts:

$$\bar{\boldsymbol{C}} \equiv \boldsymbol{C}_0 + \hat{\boldsymbol{C}}(\epsilon) \tag{19}$$

where

$$\boldsymbol{C}_0 = \begin{bmatrix} 0 & 0 & -\lambda q_e & q_e \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -C_2q_e & 0 & -C_3\sigma q_e & -\eta \end{bmatrix}, \qquad \hat{\boldsymbol{C}}(\epsilon) = \bar{\boldsymbol{C}} - \boldsymbol{C}_0 \tag{20}$$

Matrix $\boldsymbol{C}_0$ is independent of ϵ, and its eigenvalues s satisfy the characteristic equation

$$s^2\left(s^2 + \eta s + C_2q_e^2\right) = 0 \tag{21}$$

The stability criterion for the zero-order linear part of system (15), that is, $\boldsymbol{x}' = \boldsymbol{C}_0\boldsymbol{x}$, is

$$C_2 > 0, \qquad \text{or} \qquad \gamma < \pi \tag{22}$$

The sign of the discriminant $\Delta_1 (=\eta^2 - 4C_2q_e^2)$ may affect the subsequent analysis. First, the $\Delta_1 > 0$ case is considered. The eigenvalues of $\boldsymbol{C}_0$ are s_1, s_2, 0, 0, where $s_{1,2} = [-\eta \pm \sqrt{(\eta^2 - 4C_2q_e^2)}]/2$. Let the columns of the matrix $\boldsymbol{P}$ consist of the eigenvectors of $\boldsymbol{C}_0$. Introducing the transformation equation

$$\boldsymbol{x} = \boldsymbol{P}\boldsymbol{y} \tag{23}$$

and considering the parameter ϵ as a dependent variable, systems (15) are recast in the form

$$\boldsymbol{y}' = \boldsymbol{\Lambda}\boldsymbol{y} + \boldsymbol{D}(\epsilon)\boldsymbol{y} + \boldsymbol{g}(\epsilon, \boldsymbol{y}), \qquad \epsilon' = 0 \tag{24a–24e}$$

where

$$\boldsymbol{\Lambda} = \boldsymbol{P}^{-1}\boldsymbol{C}_0\boldsymbol{P} = \mathrm{diag}\{s_1, s_2, 0, 0\} \tag{25}$$

$$\boldsymbol{D}(\epsilon) = \boldsymbol{P}^{-1}\hat{\boldsymbol{C}}(\epsilon)\boldsymbol{P} = [D_{ij}], \qquad D_{ij} \in \mathcal{O}(\epsilon) \tag{26}$$

$$\boldsymbol{g}(\epsilon, \boldsymbol{y}) = \boldsymbol{P}^{-1}\boldsymbol{f}(\epsilon, \boldsymbol{P}\boldsymbol{y}) \equiv (g_1, g_2, g_3, 0)^T, \qquad g_i \in \mathcal{O}(|y|^2, \epsilon|y|^2) \tag{27}$$

Note that the terms $\boldsymbol{D}(\epsilon)\boldsymbol{y}$ in the right-hand side (RHS) of Eq. (24) are now nonlinear. The linear parts of Eqs. (24a) and (24b) have negative eigenvalues and those of Eqs. (24c–24e) have zero eigenvalues. According to the center manifold theorem[16] there exist two three-dimensional center manifolds

$$y_1 = \phi(\epsilon, y_3, y_4), \qquad y_2 = \psi(\epsilon, y_3, y_4) \tag{28}$$

The requirement that the center manifolds be invariant is satisfied by substituting Eq. (28) into Eqs. (24a) and (24b):

$$y_1' = \phi_{,y_3}y_3' + \phi_{,y_4}y_4' = s_1\phi + D_{11}\phi + D_{12}\psi + D_{14}y_4 + g_1(\epsilon, \phi, \psi, y_3, y_4) \tag{29a}$$

$$y_2' = \psi_{,y_3}y_3' + \psi_{,y_4}y_4' = s_2\psi + D_{21}\phi + D_{22}\psi + D_{24}y_4 + g_2(\epsilon, \phi, \psi, y_3, y_4) \tag{29b}$$

The substitution of Eq. (28) into Eqs. (24c) and (24d) yields the equations of the flow on the center manifolds

$$y_3' = g_3(\epsilon, \phi, \psi, y_3, y_4), \qquad y_4' = D_{41}\phi + D_{42}\psi + D_{44}y_4 \tag{30}$$

The asymptotic behavior and stability of system (30) answer the asymptotic behavior and stability of system (24). Let the second approximations for ϕ and ψ be denoted by

$$\phi = \phi^2 + \mathcal{O}[C(\epsilon, y_3, y_4)], \qquad \psi = \psi^2 + \mathcal{O}[C(\epsilon, y_3, y_4)] \tag{31}$$

where ϕ^2 and ψ^2 are homogeneous quadratics in ϵ, y_3, and y_4; $C(\epsilon, y_3, y_4)$ is a homogeneous cubic in ϵ, y_3, and y_4. Substituting Eq. (30) into Eq. (29a), by order comparison it is found that the terms $s_1\phi$, $D_{14}y_4$, and $g_1(\epsilon, \phi, \psi, y_3, y_4)$ are of order $\mathcal{O}[Q(\epsilon, y_3, y_4)]$, where $Q(\epsilon, y_3, y_4)$ represents a homogeneous quadratic in ϵ, y_3 and y_4, and the terms $\phi_{,y_3}y_3'$, $\phi_{,y_4}y_4'$, $D_{11}\phi$, and $D_{12}\psi$ are of order $\mathcal{O}[C(\epsilon, y_3, y_4)]$. By collecting terms of order $\mathcal{O}[Q(\epsilon, y_3, y_4)]$ we have

$$s_1\phi^2 + D_{14}y_4 + g_1(0, 0, 0, y_3, y_4) = 0 \tag{32}$$

where $g_1(0, 0, 0, y_3, y_4) \in \mathcal{O}(|y|^2)$, because $g_1(\epsilon, y) \in \mathcal{O}(|y|^2, \epsilon|y|^2)$. From Eq. (32) we obtain

$$\phi^2 = -(1/s_1)[D_{14}y_4 + g_1(0, 0, 0, y_3, y_4)] \tag{33a}$$

Similarly, from Eq. (29b) we obtain

$$\psi^2 = -(1/s_2)[D_{24}y_4 + g_2(0, 0, 0, y_3, y_4)] \tag{33b}$$

Substituting Eqs. (31) and (33) into Eq. (30) and using the Appendix and Eqs. (26) and (27), we have

$$y_3' = \left[(B_1\sigma + B_2k)k - B_2\sigma^2\right]y_4^2 + \mathcal{O}(\epsilon^2, |\epsilon y|^4) \tag{34a}$$

$$y_4' = \epsilon\bar{\gamma}\eta[(\sigma - 1)/\sigma]y_4 + \mathcal{O}(\epsilon^2, |\epsilon y|^4) \tag{34b}$$

From Eqs. (22) and (34) we know that if

$$\sigma < 1 \qquad \text{and} \qquad \gamma < \pi \tag{35}$$

system (24), that is, the local motion about the relative equilibrium E_3, is stable; otherwise, system (24) is unstable. It can be obtained directly from Eq. (34b) that the decay time constant of the variable y_4 that equals the quantity r is $\sigma/[\epsilon\bar{\gamma}\eta(1-\sigma)]$. From Eq. (34a) we know that the decay time constant of the variable y_3 that is equal to $q - q_e$ is $\sigma/[2\epsilon\bar{\gamma}\eta(1-\sigma)]$. The overall decay time constant τ_c will be the greater one, that is,

$$\tau_c = \frac{\sigma}{\epsilon\bar{\gamma}\eta(1-\sigma)} \tag{36}$$

For the $\Delta_1 < 0$ case, by following a similar procedure, we obtain the same stability criteria and decay time constant as in the $\Delta_1 > 0$ case. For the case of relative equilibrium E_2, the decay time constant is also the same as those of the E_3 case, but the stability criteria become[17]

$$\sigma < 1, \qquad \gamma > \pi$$

The preceding equations show that the tube must be greater than half full for stable motion. Thus, the equilibrium E_2 may not represent a practical physical reality (because the slug may split into two separate bodies because of the centrifugal effects), unless the liquid slug is assumed to have surface tension strong enough to counteract the centrifugal effects and remain intact.

Relative Equilibrium E_1 Case

The relative equilibrium point E_1 is shifted to the origin by letting $p = p_e + s_1$, $q = s_2$, $r = r_e + s_3$, and $\beta' = s_4$, where p_e and r_e satisfy Eq. (9). Substituting these into Eq. (7), we obtain

$$s' = \bar{C}s + m(s) \tag{37}$$

where the matrix $\bar{C}$ here is the same as the $\bar{C}$ in Eq. (15) and the vector $m(s)$ is of the same form as the vector $f(x)$ in Eq. (17) except that the variable x is replaced by s. The matrix $\bar{C}$ is decomposed into $\bar{C} \equiv C_0 + \hat{C}(\epsilon)$, where

$$C_0 = \begin{bmatrix} 0 & -\lambda r_e & 0 & 0 \\ \lambda r_e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -C_3\sigma r_e & 0 & -\eta \end{bmatrix} \tag{38}$$

and $\hat{C}(\epsilon) = \bar{C} - C_0$. The eigenvalues of C_0 are $\pm i\lambda r_e$, 0, and $-\eta$. The corresponding eigenvectors are $(i, 1, 0, k_2 + ik_1)^T$, $(-i, 1, 0, k_2 - ik_1)^T$, $(0, 0, 1, 0)^T$, and $(0, 0, 0, 1)^T$, where

$$k_1 = \frac{C_3\sigma r_e\omega}{\eta^2 + \omega^2}, \qquad k_2 = -\frac{C_3\sigma r_e\eta}{\eta^2 + \omega^2}, \qquad \omega = \lambda r_e \tag{39}$$

Let matrix P be composed of the eigenvectors of C_0. Through the use of the coordinates transformation equation $s = Pt$, Eq. (37) is converted into the canonical equations

$$t' = \Lambda t + D(\epsilon)t + n(\epsilon, t) \tag{40}$$

where

$$\Lambda = P^{-1}C_0P \tag{41a}$$

$$D(\epsilon) = P^{-1}\hat{C}(\epsilon)P = [D_{ij}], \qquad D_{ij} \in \mathcal{O}(\epsilon) \tag{41b}$$

and

$$n(\epsilon, t) = P^{-1}m(\epsilon, Pt) = \begin{Bmatrix} n_1 \\ n_2 \\ 0 \\ n_4 \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{4}\sum_{k=j}^{4} n_{1jk}t_jt_k \\ \sum_{j=1}^{4}\sum_{k=j}^{4} n_{2jk}t_jt_k \\ 0 \\ \sum_{j=1}^{4}\sum_{k=j}^{4} n_{4jk}t_jt_k \end{Bmatrix} \tag{41c}$$

Now we consider ϵ as a dependent variable and, hence, the second term $D(\epsilon)t$ in the RHS of Eq. (40) becomes nonlinear. We rescale the variables t by introducing $t_i = \epsilon z_i$ so that the nonlinear terms $n(\epsilon, t)$ of Eq. (40) have their leading terms of the same order as the terms $D(\epsilon)t$. Then we have

$$z_1' = -\omega z_2 + \sum_{j=1}^{2} D_{1j}(\epsilon)z_j + D_{14}(\epsilon)z_4 + \epsilon n_1(\epsilon, z) \tag{42a}$$

$$z_2' = \omega z_1 + \sum_{j=1}^{3} D_{2j}(\epsilon)z_j + D_{24}(\epsilon)z_4 + \epsilon n_2(\epsilon, z) \tag{42b}$$

$$z_3' = \sum_{j=1}^{2} D_{3j}(\epsilon)z_j + D_{34}(\epsilon)z_4 \tag{42c}$$

$$\epsilon' = 0 \tag{42d}$$

$$z_4' = -\eta z_4 + \sum_{j=1}^{3} D_{4j}(\epsilon)z_j + D_{44}(\epsilon)z_4 + \epsilon n_4(\epsilon, z) \tag{42e}$$

The eigenvalues of the linear part of Eqs. (42a–42d) have zero real parts, whereas that of Eq. (42e) has a negative real part. According to the center manifold theorem there exists one four-dimensional center manifold

$$z_4 = \Phi(\epsilon, z_1, z_2, z_3) \equiv \epsilon\phi(z_1, z_2, z_3) + \epsilon^2\varphi(z_1, z_2, z_3) + \mathcal{O}(\epsilon^3) \tag{43}$$

where $\phi, \varphi \in C^1$, and they are quadratic polynomials with $\phi(0) = \varphi(0) = 0$. Substituting Eq. (43) into Eq. (42e), we have

$$z_4' = \Phi' = \frac{\partial\Phi}{\partial z_1}z_1' + \frac{\partial\Phi}{\partial z_2}z_2' + \frac{\partial\Phi}{\partial z_3}z_3'$$
$$= -\eta\Phi + \sum_{j=1}^{3} D_{4j}z_j + D_{44}\Phi + \epsilon n_4(\epsilon, z_1, z_2, z_3, \Phi) \tag{44}$$

Substituting Eqs. (42a–42c) into Eq. (44), letting $D_{ij} = \epsilon E_{ij}$, expressing $n_{ijk} = n_{ijk}^{(0)} + \epsilon n_{ijk}^{(1)} + \epsilon^2 n_{ijk}^{(2)} + \mathcal{O}(\epsilon^3)$, collecting coefficients of equal powers of ϵ and ϵ^2, and solving the resulting equations of coefficients, the solutions of ϕ and φ are obtained as

$$\phi(z_1, z_2, z_3) = \phi_{01}z_1 + \phi_{02}z_2 + \phi_{03}z_3 + \phi_{11}z_1^2 + \phi_{12}z_1z_2 + \phi_{13}z_1z_3 + \phi_{22}z_2^2 + \phi_{23}z_2z_3 \tag{45a}$$

$$\varphi(z_1, z_2, z_3) = \frac{n_{433}^{(1)} - \phi_{23}E_{23}}{\eta}z_3^2 + \hat{\varphi}(z_1, z_2, z_3) \tag{45b}$$

where the coefficients ϕ_{ij} are functions of system parameters, and $\hat{\varphi}$ is the function that does not contain the term z_3^2.

The substitution of Eqs. (45) into Eqs. (42a–42c) yields the third-order approximate equations for the flow on the center manifold

$$z_1' = -\omega z_2 + \epsilon\left[\sum_{j=1}^{2} E_{1j}z_j + \sum_{j=1}^{3}\sum_{k=j}^{3} n_{1jk}^{(0)}z_jz_k\right] + \epsilon^2\left[E_{14}\phi + \sum_{j=1}^{3}\sum_{k=j}^{3} n_{1jk}^{(1)}z_jz_k + \sum_{j=1}^{3} n_{1j4}^{(0)}z_j\phi\right] + \epsilon^3\left[E_{14}\varphi + \sum_{j=1}^{3}\sum_{k=j}^{3} n_{1jk}^{(2)}z_jz_k + \sum_{m=1}^{3}\left(n_{1m4}^{(1)}z_m\phi + n_{1m4}^{(0)}z_m\varphi\right)\right] + \mathcal{O}(\epsilon^4) \tag{46a}$$

$$z_2' = \omega z_1 + \epsilon\left[\sum_{j=1}^{3} E_{2j}z_j + \sum_{j=1}^{3}\sum_{k=j}^{3} n_{2jk}^{(0)}z_jz_k\right] + \epsilon^2\left[E_{24}\phi + \sum_{j=1}^{3}\sum_{k=j}^{3} n_{2jk}^{(1)}z_jz_k + \sum_{j=1}^{3} n_{2j4}^{(0)}z_j\phi\right] + \epsilon^3\left[E_{24}\varphi + \sum_{j=1}^{3}\sum_{k=j}^{3} n_{2jk}^{(2)}z_jz_k + \sum_{m=1}^{3}\left(n_{2m4}^{(1)}z_m\phi + n_{2m4}^{(0)}z_m\varphi\right)\right] + \mathcal{O}(\epsilon^4) \tag{46b}$$

$$z_3' = \epsilon\sum_{j=1}^{2} E_{3j}z_j + \epsilon^2 E_{34}\phi + \epsilon^3 E_{34}\varphi + \mathcal{O}(\epsilon^4) \tag{46c}$$

Although approximate equations (46) contain the answer to the question of the stability of the original system (37), they are still strongly coupled and difficult to solve. We try to reduce the dimension of the system further by coordinate transformation and change of the independent variable as follows.

Let $z_1 = x_1\cos\theta$, $z_2 = x_1\sin\theta$, $z_3 = x_2$. Changing coordinates from (z_1, z_2, z_3) to (x_1, θ, x_2), Eq. (46) becomes

$$x_1' = \epsilon F_1(\boldsymbol{x}, \theta) + \epsilon^2 G_1(\boldsymbol{x}, \theta) + \epsilon^3 H_1(\boldsymbol{x}, \theta) + \mathcal{O}(\epsilon^4) \tag{47a}$$

$$x_2' = \epsilon F_2(\boldsymbol{x}, \theta) + \epsilon^2 G_2(\boldsymbol{x}, \theta) + \epsilon^3 H_2(\boldsymbol{x}, \theta) + \mathcal{O}(\epsilon^4) \tag{47b}$$

$$\theta' = \omega + \epsilon l_1(\boldsymbol{x}, \theta) + \epsilon^2 l_2(\boldsymbol{x}, \theta) + \mathcal{O}(\epsilon^3) \tag{47c}$$

where $F_i(\boldsymbol{x}, \theta)$, $G_i(\boldsymbol{x}, \theta)$, $H_i(\boldsymbol{x}, \theta)$, and $l_i(\boldsymbol{x}, \theta)$ are all periodic functions of θ with period 2π.

Considering θ as independent variable, Eq. (47) becomes

$$\frac{dx_i}{d\theta} = \frac{x_i'}{\theta'} = \frac{\epsilon F_i + \epsilon^2 G_i + \epsilon^3 H_i + \mathcal{O}(\epsilon^4)}{\omega + \epsilon l_1 + \epsilon^2 l_2 + \mathcal{O}(\epsilon^3)}$$
$$\equiv \epsilon f_i(\boldsymbol{x}, \theta) + \epsilon^2 g_i(\boldsymbol{x}, \theta) + \epsilon^3 h_i(\boldsymbol{x}, \theta) + \mathcal{O}(\epsilon^4) \qquad i = 1, 2 \tag{48}$$

Equation (48) is a weakly nonlinear nonautonomous system and is in the standard form with respect to averaging. A third-order near identity coordinate transformation

$$\boldsymbol{x} = \boldsymbol{y} + \epsilon\boldsymbol{u}(\boldsymbol{y}, \theta) + \epsilon^2\boldsymbol{v}(\boldsymbol{y}, \theta) + \epsilon^3\boldsymbol{w}(\boldsymbol{y}, \theta) \tag{49}$$

is introduced without yet choosing $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ to convert the nonautonomous system (48) to an autonomous one by the method of averaging.[15,18] The differentiation of Eq. (49) with respect to θ yields

$$\frac{d\boldsymbol{y}}{d\theta} = \left(\boldsymbol{I} + \epsilon\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{y}} + \epsilon^2\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{y}} + \epsilon^3\frac{\partial\boldsymbol{w}}{\partial\boldsymbol{y}}\right)^{-1} \times\left(\frac{d\boldsymbol{x}}{d\theta} - \epsilon\frac{\partial\boldsymbol{u}}{\partial\theta} - \epsilon^2\frac{\partial\boldsymbol{v}}{\partial\theta} - \epsilon^3\frac{\partial\boldsymbol{w}}{\partial\theta}\right) + \mathcal{O}(\epsilon^4) \tag{50}$$

Substituting Eq. (49) into Eq. (48) and expanding the resulting equation in powers of ϵ up to the order ϵ^3, we have

$$\frac{d\boldsymbol{x}}{d\theta} = \epsilon\boldsymbol{f} + \epsilon^2\left(\boldsymbol{g} + \frac{\partial\boldsymbol{f}}{\partial\boldsymbol{y}}\boldsymbol{u}\right) + \epsilon^3\left(\boldsymbol{h} + \frac{\partial\boldsymbol{f}}{\partial\boldsymbol{y}}\boldsymbol{v} + \frac{\partial\boldsymbol{g}}{\partial\boldsymbol{y}}\boldsymbol{u} + \frac{1}{2}\frac{\partial^2\boldsymbol{f}}{\partial\boldsymbol{y}\partial\boldsymbol{y}}\boldsymbol{u}\boldsymbol{u}\right) + \mathcal{O}(\epsilon^4) \tag{51}$$

If $\mathcal{F}(\boldsymbol{y}, \theta)$ is the periodic function of θ with period T, let $\langle\mathcal{F}\rangle$ denote the average of the function $\mathcal{F}$ over the period T, i.e.,

$$\langle\mathcal{F}\rangle = \frac{1}{T}\int_0^T \mathcal{F}\,d\theta$$

Define $\hat{\boldsymbol{g}}$ and $\hat{\boldsymbol{h}}$ as

$$\hat{\boldsymbol{g}} \equiv \boldsymbol{g} + \frac{\partial\boldsymbol{f}}{\partial\boldsymbol{y}}\boldsymbol{u} - \frac{\partial\boldsymbol{u}}{\partial\boldsymbol{y}}\langle\boldsymbol{f}\rangle \tag{52}$$

$$\hat{\boldsymbol{h}} \equiv \boldsymbol{h} + \frac{\partial\boldsymbol{f}}{\partial\boldsymbol{y}}\boldsymbol{v} + \frac{\partial\boldsymbol{g}}{\partial\boldsymbol{y}}\boldsymbol{u} + \frac{1}{2}\frac{\partial^2\boldsymbol{f}}{\partial\boldsymbol{y}\partial\boldsymbol{y}}\boldsymbol{u}\boldsymbol{u} - \frac{\partial\boldsymbol{u}}{\partial\boldsymbol{y}}\langle\hat{\boldsymbol{g}}\rangle - \frac{\partial\boldsymbol{v}}{\partial\boldsymbol{y}}\langle\boldsymbol{f}\rangle \tag{53}$$

Let $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ be chosen as the solutions of the following equations:

$$\frac{\partial\boldsymbol{u}}{\partial\theta} = \boldsymbol{f} - \langle\boldsymbol{f}\rangle, \qquad \frac{\partial\boldsymbol{v}}{\partial\theta} = \hat{\boldsymbol{g}} - \langle\hat{\boldsymbol{g}}\rangle, \qquad \frac{\partial\boldsymbol{w}}{\partial\theta} = \hat{\boldsymbol{h}} - \langle\hat{\boldsymbol{h}}\rangle \tag{54}$$

The arbitrary constants of integration in Eq. (54) are chosen such that $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ have zero means, that is, $\langle\boldsymbol{u}\rangle = 0$, $\langle\boldsymbol{v}\rangle = 0$, and $\langle\boldsymbol{w}\rangle = 0$. Expanding Eq. (50) and using Eqs. (51–54), we have

$$\frac{d\boldsymbol{y}}{d\theta} = \epsilon\langle\boldsymbol{f}\rangle + \epsilon^2\langle\hat{\boldsymbol{g}}\rangle + \epsilon^3\langle\hat{\boldsymbol{h}}\rangle + \mathcal{O}(\epsilon^4) \tag{55}$$

Solving Eq. (54) for the averaged terms in Eq. (55) and introducing the new independent variable $\vartheta = \theta/\omega$, Eq. (55) becomes

$$\begin{aligned}\frac{\mathrm{d}y_1}{\mathrm{d}\vartheta} &= \epsilon\hat{a}_1 y_1 + \epsilon^2 a_{12} y_1 y_2 + \epsilon^3 \bar{h}_1(y) + \mathcal{O}(\epsilon^4) \\ \frac{\mathrm{d}y_2}{\mathrm{d}\vartheta} &= \epsilon^2 b_{11} y_1^2 + \epsilon^3 b_{22} y_2^2 + \epsilon^3 \bar{h}_1(y) + \mathcal{O}(\epsilon^4)\end{aligned} \tag{56}$$

where $\bar{h}_1(y)$ and $\bar{h}_2(y)$ are nonlinear terms and the coefficients $a_1, a_{10}, a_{12}, b_{11}$, and b_{22} are function of system parameters such that

$$a_1 = \frac{E_{11} + E_{22}}{2}$$

$$a_{10} = \tfrac{1}{2}[\phi_{01}E_{14} + \phi_{02}E_{24} + (1/2\omega)(E_{11} + E_{22}) \times (E_{12} - E_{21}) + (E_{23}E_{31}/2\omega)] \tag{57}$$

and

$$\hat{a}_1 = a_1 + \epsilon a_{10} \tag{58}$$

If the first-order averaged system is considered, that is, only the terms of order ϵ^1 in Eq. (56) are retained, then Eq. (56) reduces to

$$\frac{\mathrm{d}y_1}{\mathrm{d}\vartheta} = \epsilon\hat{a}_1 y_1 + \mathcal{O}(\epsilon^2) \tag{59a}$$

$$\frac{\mathrm{d}y_2}{\mathrm{d}\vartheta} = 0 + \mathcal{O}(\epsilon^2) \tag{59b}$$

It is obvious that even though $a_1 < 0$, Eq. (59) is inconclusive for the stability question since Eq. (59b) has zero eigenvalue. It is also found that the second-order averaged system fails for the determination of stability,[17] so third-order averaged equations are required.

System (56) is unstable if $\hat{a}_1 > 0$, and if $\hat{a}_1 < 0$, there exists a center manifold $y_1 = \chi(y_2)$ such that the asymptotic behavior of the system (56) is determined by

$$\frac{\mathrm{d}y_2}{\mathrm{d}\vartheta} = \epsilon^2 b_{11}\chi^2(y_2) + \epsilon^3 b_{22} y_2^2 + \epsilon^3 \bar{h}_2[\chi(y_2), y_2] + \mathcal{O}(\epsilon^4) \tag{60}$$

Substituting $y_1 = \chi(y_2)$ into Eq. (56b), $\chi(y_2)$ is obtained in the form

$$\chi(y_2) = -\epsilon^2(\bar{h}_{102}/\hat{a}_1)y_2^2 + \mathcal{O}(|y_2|^3) \tag{61}$$

where $\bar{h}_{102}$ is the coefficient of the y_2^2 term of $\bar{h}_1(y)$. Substituting Eq. (61) into Eq. (60) yields

$$\frac{\mathrm{d}y_2}{\mathrm{d}\vartheta} = \epsilon^3 b_{22} y_2^2 + \mathcal{O}(|y_2|^3) \tag{62}$$

Note that the quantities $\bar{h}_1(y)$ and $\bar{h}_2(y)$ that do not contain the term y_2^2 do not appear in Eq. (62). Thus, they are not elaborated in Eq. (56). The solution of Eq. (62) is

$$y_2(\vartheta) = \frac{1}{[1/y_2(0)] - \epsilon^3 b_{22}\vartheta} \tag{63}$$

Since $b_{22} = E_{31}E_{23}2\lambda/\omega^2 = 2\bar{\gamma}^2\kappa^2 b^2 r_e \eta/(\eta^2 + \omega^2) > 0$, from Eq. (63) we know that system (62) is stable if $y_2(0) < 0$ and is unstable if $y_2(0) > 0$. It is proved in Ref. 17 that $y_2(0) < 0$. Thus, by the center manifold theorem and the theory of the method of averaging, the local motion behavior of the system (7) about E_1 is stable if

$$\hat{a}_1 = a_1 + \epsilon a_{10} < 0 \tag{64}$$

and unstable if $\hat{a}_1 > 0$. Substituting Eq. (57) into Eq. (58) for a_1 and a_{10} and using Eqs. (12), (39), and the Appendix, Eq. (64) becomes

$$\hat{a}_1 = -\frac{\bar{\gamma}\kappa^2 b^2 \sigma^3 \eta r_e^2}{2(\sigma - 1)\left[\eta^2 + (\sigma - 1)^2 r_e^2\right]} + \epsilon\mathcal{G}_0(\sigma, r_e, \bar{\gamma}, b, \eta) + \mathcal{O}(\epsilon^2) < 0 \tag{65}$$

where $\mathcal{G}_0$ is the function of the parameters of the damper and the spacecraft. Equation (65) shows that the stability criterion involves the parameters of the nutation damper. If first approximation is considered, that is, only the ϵ^0 term in the RHS of Eq. (65) is taken into account, the criterion for stable motion becomes

$$\sigma > 1 \tag{66}$$

and the motion is unstable if $\sigma < 1$. The corresponding physical meaning is that for a spin spacecraft carrying a viscous ring damper, the motion about the relative equilibrium E_1 is stable only if the spacecraft spins about the principal axis of maximum moment of inertia.

For practical use, designing the system parameters such that the wobble motion will decay most rapidly (in other words, the system will have minimum decay time constant) is of great importance. The decay time constant τ_c^* of the linearized equation of system (56) is $\tau_c^* = 1/(a_1 + \epsilon a_{10})$. Using the definition $\vartheta = \theta/\omega$ [Eq. (47c)] and the equation $t_i = \epsilon z_i$, the decay time constant τ_c of the system (7) that is approximated to order ϵ^{-1} is

$$\tau_c \approx \frac{1}{\epsilon a_1} \approx -\frac{2(\sigma - 1)\left[\eta^2 + (\sigma - 1)^2 r_e^2\right]}{\epsilon\bar{\gamma}\kappa^2 b^2 \sigma^3 \eta r_e^2} \tag{67}$$

Results and Discussion

An example in which the dimensionless parameters are $\epsilon = 0.005, b = 1, \sigma = 0.8, \gamma = 100$ deg, and $\eta = 0.21$ is considered. Since $\sigma < 1$ and $\gamma < \pi$, according to analytical solution (35) the motion in the neighborhood of the equilibrium E_3 is stable. This result is verified by the solution for the r component of angular velocity, as shown in Fig. 2, that is obtained by integrating numerically the original nonlinear equations (7) and considered as exact. It starts from the initial state $(p_0, q_0, r_0) = (1.0, 0.1, 0.1)$, which is

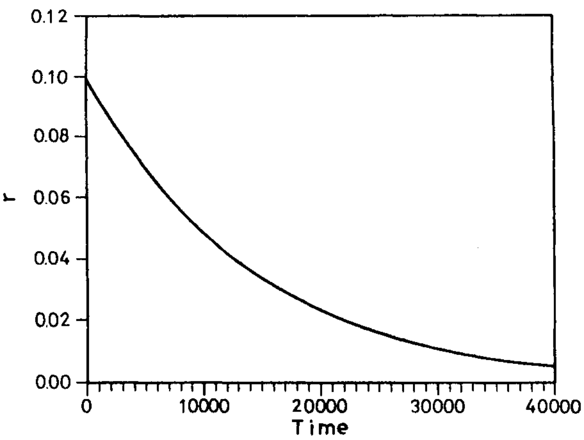

Fig. 2 Time history of the r component of the spacecraft angular velocity with $(p_0, q_0, r_0)^T = (1.0, 0.1, 0.1)^T$: time constant = 13,782 (numerical solution) and 13,720 (analytical solution).

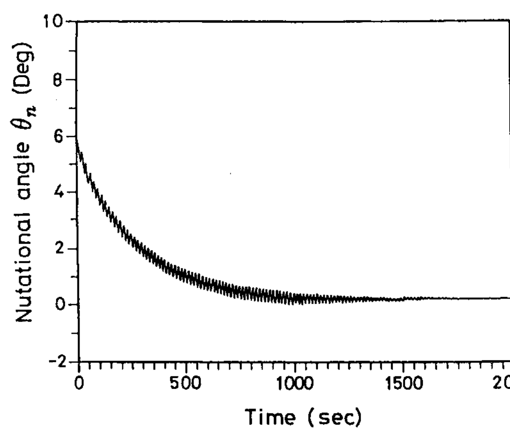

Fig. 3 Time history of the nutation angle for the E_1 case: $\tau_c = 322$ s.

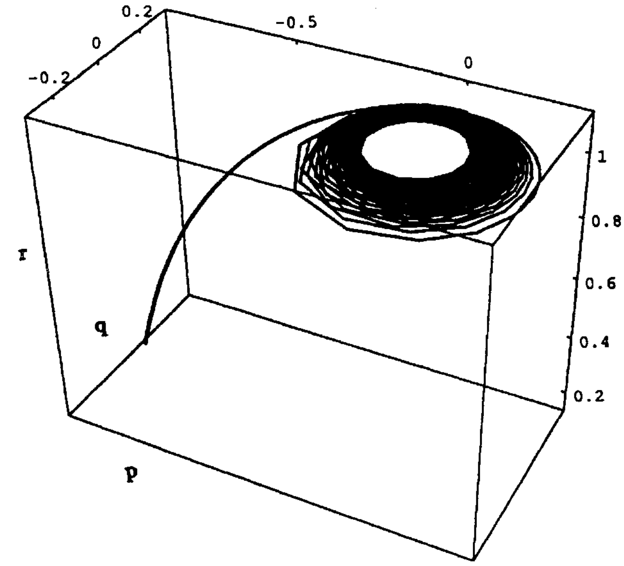

Fig. 4 Three-dimensional phase diagram with $(p_0, q_0, r_0)^T$ = $(0.1, 0.1, 1, 1)^T$; diverge from E_1 and converge to E_2.

in the vicinity of E_2 and then approaches the equilibrium state E_3 where $r_e = 0$, $p_e = 0$, and $q_e \neq 0$, which agrees well with analytical expression (11). The decay-time constant of the r component in Fig. 2 by curve fitting is 13,782, which is close to the analytical result 13,720. The numerical solution for the time constant of the q component is 6896, which almost matches the predicted value, i.e., one-half of the r component of Eq. (34). Note that the analytical expression for the decay-time constant, i.e., Eq. (36), is independent of b (the nondimensional distance of the damper to the center of mass of the spacecraft), and the decay-time constant increases as the inertia ratio σ $(=C/A)$ approaches the value 1.

If we choose $\sigma = 1.4$ and $\gamma = 200$ deg with other parameters unchanged, the numerical solution for the motion, with initial conditions in the neighborhood of the equilibrium E_1, is stable. The decay-time constant of the nutation angle delineated in Fig. 3 is 322 by curve fitting, which is close to the value 320 predicted by analytical equation (67). Note that the nutation angle of the E_1 case exhibits oscillating decay but that of the E_3 case decays monotonously, as shown in Fig. 2. Equation (67) shows that the decay-time constant τ_c is function of r_e (which is the ratio of the spin rate at the equilibrium state to that at the initial state); therefore, the decay-time constant for the E_1 case depends on the spin rate of the equilibrium state but it does not for either the E_3 or E_2 case. If $r_e = 1$, our result is the same as that of Alfriend.[10] Since Alfriend used zero-order approximation to obtain the solution of spacecraft motion, he got the result of constant spin rate. This can be seen by noting that Eq. (7d), $r' = C_4\beta'/\Omega = \epsilon\bar{\gamma}\eta/\sigma\Omega$, reduces to $r' = 0$ if zero-order approximation is assumed; this results in $r_e = 1$. The decay-time constant is inversely proportional to the square of the parameter b, whereas the time constant for the the E_3 (or E_2) case has nothing to do with the parameter b. Equation (67) shows that τ_c decreases monotonously as σ approaches 1, but the time constant for the E_3 (or E_2) case increases as σ tends to the value 1. Equation (67) may lead us to the incorrect conclusion that the time constant will go to zero as σ approaches 1. Note that Eq. (67) is valid only for $\sigma - 1 \neq \mathcal{O}(\epsilon)$, which is assumed in the beginning of the analysis. The numerical solution of Eq. (7) reveals that the time constant will go to infinity as σ tends to the value 1. Thus, we conjecture that in the range $0 < \sigma < 1 + \mathcal{O}(\epsilon)$, the time constant is a concave function of σ and has a minimum. The stability questions and decay-time constant for σ in that range are still unsolved.

The global analysis is treated by numerical computations. Those cases where the system loses its stability about one equilibrium point as the system parameters cross the stability boundaries and bifurcates into another stable solution are studied here. With system parameters given as $\sigma = 0.8$ and $\gamma = 300$ deg, even if the initial point lies in the neighborhood of E_1, the solution diverges away from E_1 and converges to E_2 [where $r_e = \mathcal{O}(\epsilon)$ and $p_e = \mathcal{O}(1)$], as shown in Fig. 4, because the system parameters satisfy the stability criterion of E_2, not of E_1. Figure 5 shows that for the case where $\sigma = 1.4$ and $\gamma = 300$ deg, the system with an initial point in the neighborhood of E_3 diverges away from E_3 and converges to E_1.

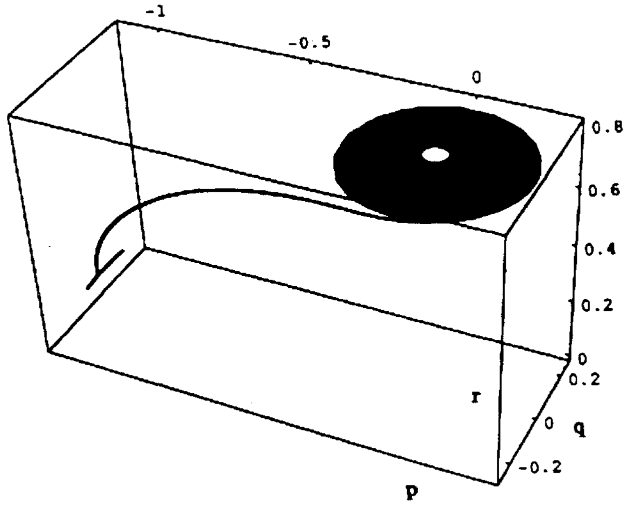

Fig. 5 Three-dimensional phase diagram with $(p_0, q_0, r_0)^T$ = $(-1.1, 0.1, 0.1)^T$; diverge from E_3 and converge to E_1.

Conclusions

The dynamics and stability of a freely precessing spacecraft containing a viscous ring nutation damper are analyzed by analytical techniques. There exist three relative equilibria denoted by E_1, E_2, and E_3 for various values of system parameters. The spacecraft spins about the inertially symmetric axis for the E_1 case but spins about the transverse axis for the E_2 and E_3 cases. The stability criteria and decay-time constant for E_3 (or E_2) are newly found in this paper. The effects of system parameters on the local motion behavior about E_3 (or E_3) are quite different from that about E_1; for example, the decay-time constants for E_2 and E_3 are independent of the spin rate at the equilibrium state and the distance of the damper to the mass center of the spacecraft, but these are entirely opposite to the E_1 case. The global analyses are performed numerically and verify the results of local bifurcation analysis.

Appendix: Equation (7) Coefficients

The coefficients of Eq. (7) are

$$\lambda = \sigma - 1, \qquad \kappa = \frac{\sin(\gamma/2)}{\gamma/2}, \qquad \bar{\gamma} = \gamma/2\pi$$

$$A_1 = \epsilon\bar{\gamma}\{\kappa^2 b^2 - b^2 + [1 - (\sin\gamma/\gamma)]/2\}$$

$$A_2 = \epsilon\bar{\gamma}\kappa b[(\sin\gamma/\gamma) - 1], \qquad A_3/\Omega = \epsilon\bar{\gamma}\kappa b\eta$$

$$B_1 = \epsilon\bar{\gamma}\{-b^2 + [1 + (\sin\gamma/\gamma)]/2\}, \qquad B_2 = -\epsilon\bar{\gamma}\kappa b$$

$$C_1/\Omega = [1 + \epsilon(\bar{\gamma}/\sigma)]\eta = \eta + C_4/\Omega, \qquad C_2 = \sin\gamma/\gamma$$

$$C_3 = \kappa b, \qquad C_4/\Omega = \epsilon\bar{\gamma}(\eta/\sigma)$$

$$D_1 = 1 + \epsilon\bar{\gamma}\{b^2 - \kappa^2 b^2 + [1 - (\sin\gamma/\gamma)]/2\}$$

$$D_2 = 1 + \epsilon\bar{\gamma}\{b^2 + [1 + (\sin\gamma/\gamma)]/2\}$$

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